MIDESMESTRAL EXAMINATION
M.Math. I - March 6, 2006
Algebra II - B.Sury

Maximum Marks 75; Any score > 75 will be equated to 75.
NOT ALL questions carry equal marks.
Answer FOUR questions from section A, THREE from section B.
Be BRIEF. Quote clearly the results you use without proof.

SECTION A

A 1. (8 marks)
If $q = p^n$ and $\alpha \in \mathbf{F}_q$, show that

$$(X-\alpha)(X-\alpha^p)(X-\alpha^{p^2})\cdots(X-\alpha^{p^{n-1}}) \in \mathbf{F}_p[X].$$

A 2. (10 marks)
Let $F = Spl(X^{13}-1, \mathbf{F}_3)$. Prove that $[F : \mathbf{F}_3] = 3$.

A 3. (12 marks)
(a) If $\mathbf{Q}(\alpha)$ and $\mathbf{Q}(\beta)$ are extensions of degrees m, n over $\mathbf{Q}$ which are relatively prime, prove that $min(\alpha, \mathbf{Q})$ remains irreducible in $\mathbf{Q}(\beta)[X]$.
(b) Using (a), determine the number of irreducible factors of $1+X+\cdots+X^{p-1}$ in $\mathbf{Q}(2^{1/n})$ where $n \geq p$ are both primes.

A 4. (10 marks)
Let n be any natural number. Determine the degree of $\mathbf{Q}(Cos\frac{2\pi}{n})$ over $\mathbf{Q}$. Is this a Galois extension of $\mathbf{Q}$? Justify. Find all the conjugates of $Cos\frac{2\pi}{n}$ over $\mathbf{Q}$.

A 5. (12 marks)
If $n > 1$ is odd, then prove that $\mathbf{Q}(\zeta_d)$ cannot contain an n-th root of 2 for any d.

A 6. (8 marks)
Let E/F be an extensions of finite fields. Prove that the norm map from E to F is surjective.

A 7. (12 marks)
(a) Show that $\alpha = \sqrt{2-\sqrt{2}} + i\sqrt{\sqrt{2}-1}$ is an algebraic number (that is, a

complex number which is algebraic over $\mathbf{Q}$) with the property $|\alpha| = 1$ but that $|\beta| \neq 1$ for some conjugate β of α.
(b) Give an example of an algebraic number α such that α as well as its conjugates have absolute value 1 but are not roots of unity.

SECTION B

B 1. (10 marks)
Describe (with brief explanations) the Galois group of $K = \mathbf{Q}(\sqrt{p_1}, \cdots, \sqrt{p_n})$ over $\mathbf{Q}$ where $p_1, \cdots, p_n$ are distinct primes. Compute, using the fundamental theorem of Galois theory, the number of intermediate fields between $\mathbf{Q}$ and K.

B 2. (10 marks)
Determine all the natural numbers n for which the angle of n° can be constructed using a ruler and a compass.

B 3. (10 marks)
Let n be a natural number and Φ_n be the minimal polynomial of $e^{2i\pi/n}$ over $\mathbf{Q}$; that is, the cyclotomic polynomial of degree $\phi(n)$. For any integer a, show that any prime factor p of $\Phi_n(a)$ with $p \not| n$ must satisfy $p \equiv 1 \bmod n$.

B 4. (8 marks)
Let Char. $K = p > 0$, and let $a \in K$. If the polynomial $X^p - X - a$ is reducible in $K[X]$, prove that all its roots lie in K.

B 5. (8 marks)
Let Char. $K = p > 0$. Suppose L/K is a finite extension such that $p \not| [L : K]$. Show that L/K is separable.

B 6. (12 marks)
Prove that $K = Spl(X^3 - 3X + 1, \mathbf{Q})$ is not a radical extension of $\mathbf{Q}$.